Averaging operators over nondegenerate quadratic surfaces in finite fields

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ABSTRACT. We study mapping properties of the averaging operator related to the variety $V = \{x \in \mathbb{F}_q^d : Q(x) = 0\}$, where $Q(x)$ is a nondegenerate quadratic polynomial over a finite field $\mathbb{F}_q$ with q elements. In the previous paper [**4**], sharp $L^p - L^r$ averaging estimates were given in all odd dimensions $d \geq 3$ but not in even dimensions $d \geq 4$. In this paper we investigate the sharp strong-type estimates of averaging operators over varieties V in even dimensions $d \geq 4$. Critical endpoint estimates are successfully carried out to improve upon the previously known results in [**4**]. Our results are sharp in the sense that a further improvement could not be expected if V contains a $d/2$-dimensional subspace and the dimensions $d \geq 4$ are even.

1. Introduction

Let $V \subset \mathbb{R}^d$ be a smooth hypersurface and $d\sigma$ a smooth, compactly supported surface measure on V. An averaging operator A over V is given by

$$Af(x) = f * d\sigma(x) = \int_V f(x-y)d\sigma(y)$$

where f is a complex valued function on $\mathbb{R}^d$. In this Euclidean setting, the averaging problem is to determine the optimal range of exponents $1 \leq p, r \leq \infty$ such that

$$\|f * d\sigma\|_{L^r(\mathbb{R}^d)} \leq C_{p,r,d}\|f\|_{L^p(\mathbb{R}^d)}, \ f \in \mathcal{S}(\mathbb{R}^d) \tag{1.1}$$

where $\mathcal{S}(\mathbb{R}^d)$ denotes the space of Schwartz functions. When V is the unit sphere $\mathbb{S}^{d-1}$, this problem is closely related to regularity estimates of the solutions to the wave equation at time $t = 1$, and it was studied by R.S. Strichartz [**10**]. It is well known that $L^p - L^r$ averaging results can be obtained by the decay estimates of the Fourier transform of the surface measure $d\sigma$ on V. For instance, if $|\widehat{d\sigma}(\xi)| = \left|\int_V e^{-2\pi i x\cdot\xi} d\sigma(x)\right| \lesssim (1+|\xi|)^{-\alpha}$ for some $\alpha > 0$, then the averaging inequality (1.1) holds whenever

$$1 \leq p \leq 2, \quad \frac{1}{p} - \frac{1}{2} \leq \frac{1}{2}\left(\frac{\alpha}{\alpha+1}\right), \text{ and } r = p',$$

where p' denotes the exponent conjugate to p (see [**10**], [**5**], and P. 371 in [**9**]). Thus, if $|\widehat{d\sigma}(\xi)| \lesssim (1+|\xi|)^{-(d-1)/2}$ and $(1/p, 1/r) = (d/(d+1), 1/(d+1))$, then the averaging estimate (1.1) holds. Since $L^1 - L^1$ and $L^\infty - L^\infty$ estimates are clearly possible, we see from the interpolation theorem that if $|\widehat{d\sigma}(\xi)| \lesssim (1+|\xi|)^{-(d-1)/2}$, then $L^p - L^r$ estimates hold whenever $(1/p, 1/r)$ lies in the triangle Δ_d with vertices $(0,0), (1,1)$, and $(d/(d+1), 1/(d+1))$. Moreover, it is well known that $L^p - L^r$ estimates are impossible if $(1/p, 1/r)$ lies outside of the triangle Δ_d. Such analogous phenomena

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were also observed in the finite field setting (see [**1**] and [**4**]).
On the other hand, if the optimal Fourier decay estimate of the surface measure $d\sigma$ is given by

$$|\widehat{d\sigma}(\xi)| \lesssim (1+|\xi|)^{-\alpha} \quad \text{for some } \alpha < (d-1)/2,$$

then it is in general hard to prove sharp averaging results and some technical arguments are required to deal with the case (e.g. see [**3**]). In the finite field case, this was also pointed out by the authors in [**1**].

As an analogue of Euclidean averaging problems, Carbery, Stones, and Wright [**1**] initially introduced and studied the averaging problem in finite fields. Let $\mathbb{F}_q^d$ be a d-dimensional vector space over a finite field $\mathbb{F}_q$ with q elements. We endow the space $\mathbb{F}_q^d$ with a normalized counting measure "dx". Given a function $f : (\mathbb{F}_q^d, dx) \to \mathbb{C}$, its integral is given by

$$\int_{\mathbb{F}_q^d} f(x)\ dx = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} f(x).$$

Let $V \subset \mathbb{F}_q^d$ be an algebraic variety. Then, a normalized surface measure σ supported on V can be defined by the relation

$$\int f(x)\ d\sigma(x) = \frac{1}{|V|} \sum_{x \in V} f(x)$$

where $|V|$ denotes the cardinality of V (see [**8**]). Notice that the normalized surface measure σ on V can be viewed as a function on $(\mathbb{F}_q^d, dx)$:

$$\sigma(x) = \frac{q^d}{|V|} V(x).$$

Here, and throughout the paper, we write $V(x)$ for a characteristic function on V. Then, an averaging operator A can be defined by

$$Af(x) = f * d\sigma(x) = \int f(x-y)\ d\sigma(y) = \frac{1}{|V|} \sum_{y \in V} f(x-y)$$

where both f and Af are functions on $(\mathbb{F}_q^d, dx)$. In this setting the averaging problem over V is to determine $1 \le p, r \le \infty$ such that

$$\|Af\|_{L^r(\mathbb{F}_q^d, dx)} \le C \|f\|_{L^p(\mathbb{F}_q^d, dx)}, \tag{1.2}$$

where the constant $C > 0$ is independent of functions f and q, the cardinality of the underlying finite field $\mathbb{F}_q$.

Definition 1.1. Let $1 \le p, r \le \infty$. We denote by $A(p \to r) \lesssim 1$ to indicate that the averaging inequality (1.2) holds.

The main purpose of this paper is to obtain the complete $L^p - L^r$ estimates of the averaging operators over varieties determined by nondegenerate quadratic form over $\mathbb{F}_q$. Let $Q(x) \in \mathbb{F}_q[x_1, \dots, x_d]$ be a nondegenerate quadratic form. Define a variety

$$S = \{x \in \mathbb{F}_q^d : Q(x) = 0\}.$$

We shall name this kind of varieties as a nondegenerate quadratic surface in $\mathbb{F}_q^d$. Since $Q(x)$ is a nondegenerate quadratic form, it can be transformed into a diagonal form $a_1 x_1^2 + \cdots + a_d x_d^2$ with $a_j \neq 0$ by means of a linear substitution (see [**7**]). Therefore we may assume that any nondegenerate quadratic surface can be written by the form

$$S = \{x \in \mathbb{F}_q^d : a_1 x_1^2 + \cdots + a_d x_d^2 = 0\} \tag{1.3}$$

where $a_j \in \mathbb{F}_q \setminus \{0\}, j = 1, \dots, d$.

1.1. Necessary conditions for $A(p \to r) \lesssim 1$. We recall the necessary conditions for the averaging estimates over the nondegenerate quadratic surface S. It is well known from [**7**] that $|S| \sim q^{d-1}$ for $d \geq 4$ (see also Corollary 2.2 of this paper for an alternative proof). Here, $A \lesssim B$ indicates that there exists $C > 0$ independent of $q = |\mathbb{F}_q|$ such that $A \leq CB$, and $A \sim B$ means that $A \lesssim B$ and $B \lesssim A$. Let $1 \leq p, r \leq \infty$. In order to find necessary conditions for $A(p \to r) \lesssim 1$, let us suppose that the averaging estimate holds:

$$\|f * d\sigma\|_{L^r(\mathbb{F}_q^d, dx)} \lesssim \|f\|_{L^p(\mathbb{F}_q^d, dx)}. \tag{1.4}$$

It follows from a direct computation that if $f = \delta_0$, then we must have

$$\frac{d}{p} \leq \frac{1}{r} + d - 1, \tag{1.5}$$

where $\delta_0(x) = 1$ for $x = (0, \dots, 0)$ and $\delta_0(x) = 0$ otherwise. By duality, we also see tat

$$\frac{d}{r'} \leq \frac{1}{p'} + d - 1. \tag{1.6}$$

Combining this condition with (1.5) yields that $A(p \to r) \lesssim 1$ only if $(1/p, 1/r)$ lies in the convex hull of $(0,0), (0,1), (1,1)$, and $(d/(d+1), 1/(d+1))$. The authors in [**4**] proved that this necessary conditions for $A(p \to r) \lesssim 1$ are in fact the sufficient conditions for it if the dimension $d \geq 3$ is odd. An interesting fact is that this statement could not be true any more if $d \geq 4$ is even and the variety S contains a $d/2$-dimensional subspace. In this case, the necessary conditions for $A(p \to r) \lesssim 1$ can be improved. To see this, let H be a subspace with $|H| = q^{d/2}$. In addition, assume that $H \subset S$. Since H is a subspace, it is not hard to see that $H * d\sigma(x) = |H|/|S|$ for $x \in H$. Since

$$\|H * d\sigma\|_{L^r(\mathbb{F}_q^d, dx)} \geq \left(q^{-d} \sum_{x \in H} |H * d\sigma(x)|^r \right)^{1/r} \sim q^{(2-d)/2 - d/2r},$$

it must follow from (1.4) that

$$\frac{1}{p} + \frac{2-d}{d} \leq \frac{1}{r}.$$

In conclusion, if $d \geq 4$ is even and S contains a subspace H with $|H| = q^{d/2}$, then $A(p \to r) \lesssim 1$ only if $(1/p, 1/r)$ lies in the convex hull of

$$(0,0), (0,1), (1,1), \left(\frac{d^2 - 2d + 2}{d(d-1)}, \frac{1}{d-1} \right), \text{ and } \left(\frac{d-2}{d-1}, \frac{d-2}{d(d-1)} \right). \tag{1.7}$$

In this paper we show that (1.7) gives the necessary and sufficient conditions for $A(p \to r) \lesssim 1$ in the specific case when the variety S contains $d/2$-dimensional subspace with $d \geq 4$ even. See Figure 1 below.

1.2. Statement of main result.

Theorem 1.2. *Let $d\sigma$ be the normalized surface measure on the nondegenerate quadratic surface $S \subset \mathbb{F}_q^d$, as defined in (1.3). Suppose that $d \geq 4$ is an even integer and S contains a $d/2$-dimensional subspace. Then $A(p \to r) \lesssim 1$ if and only if $(1/p, 1/r)$ lies in the convex hull of*

$$(0,0), (0,1), (1,1), \left(\frac{d^2 - 2d + 2}{d(d-1)}, \frac{1}{d-1} \right), \textit{ and } \left(\frac{d-2}{d-1}, \frac{d-2}{d(d-1)} \right).$$

Remark 1.3. If the dimension $d \geq 4$ is even, then there exists a nondegenerate quadratic surface S which contains a $d/2$-dimensional subspace H. For example, consider $S = \{x \in \mathbb{F}_q^d : \sum_{k=1}^{d} (-1)^{k+1} x_k^2 = 0\}$ and $H = \{(t_1, t_1, t_2, t_2, \dots, t_{d/2}, t_{d/2}) \in \mathbb{F}_q^d : t_1, t_2, \dots, t_{d/2} \in \mathbb{F}_q\}$.

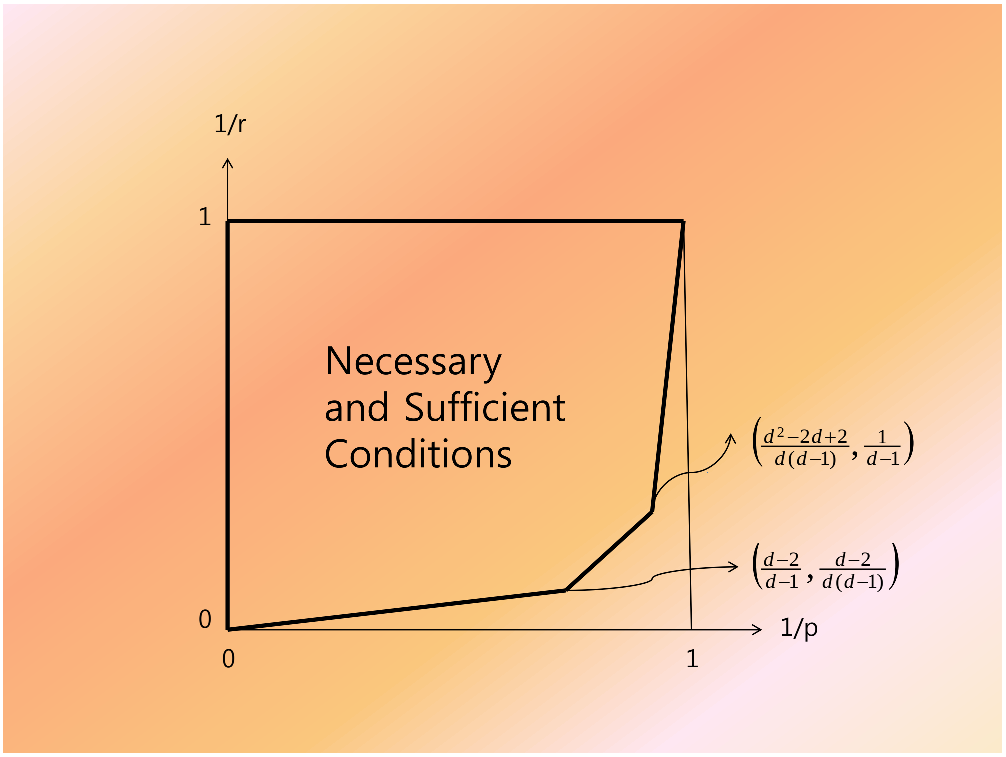

FIGURE 1. Necessary and Sufficient Conditions for $A(p \to r) \lesssim 1$ in the case that $d \geq 4$ is even and S contains a $d/2$-dimensional subspace

REMARK 1.4. From the observation (1.7), we only need to prove the " if " part of Theorem 1.2. Since dx is the normalized counting measure on $\mathbb{F}_q^d$, it follows from Young's inequality that $A(p \to r) \lesssim 1$ for $1 \leq r \leq p \leq \infty$. Thus, by duality and the interpolation theorem, it will be enough to prove that

$$\|f * d\sigma\|_{L^{d-1}(\mathbb{F}_q^d, dx)} \lesssim \|f\|_{L^{d(d-1)/(d^2-2d+2)}(\mathbb{F}_q^d, dx)} \text{ for all functions } f \text{ on } \mathbb{F}_q^d. \tag{1.8}$$

The authors in [**4**] showed that this inequality holds for all characteristic functions on subsets of $\mathbb{F}_q^d$. Here, we improve upon their work by obtaining the strong-type estimate.

1.3. Outline of the paper. In the remaining parts of this paper, we focus on providing the detail proof of Theorem 1.2. In Section 2, we review the Fourier analysis in finite fields and prove key lemmas which are essential in proving our main theorem. The proof of Theorem 1.2 for even dimensions $d \geq 6$ will be completed in Section 3. Namely, when $d \geq 6$ is any even integer, the inequality (1.8) will be proved in Section 3. In the final section, we finish the proof of Theorem 1.2 by proving the inequality (1.8) for $d = 4$.

2. Discrete Fourier analysis and key lemmas

In this section we drive key lemmas which play a crucial role in proving Theorem 1.2. We begin by reviewing the Discrete Fourier analysis. Let $\mathbb{F}_q$ be a finite field with q elements. Here, and throughout the paper, we assume that q is a power of odd prime so that the characteristic of $\mathbb{F}_q$ is greater than two. Now, let us review the definition of the canonical additive character of $\mathbb{F}_q$. Let $q = p^s$ with p prime. Recall that the trace function $Tr : \mathbb{F}_q \to \mathbb{F}_p$ is defined by

$$Tr(c) = c + c^p + \cdots + c^{p^{s-1}} \quad \text{for } c \in \mathbb{F}_q.$$

We identify $\mathbb{F}_p$ with $\mathbb{Z}/(p)$. Then the function χ defined by $\chi(a) = e^{2\pi i Tr(a)/p}$ for all $a \in \mathbb{F}_q$ is called the canonical additive character of $\mathbb{F}_q$. For example, if q is prime, then $\chi(s) = e^{2\pi i s/q}$. Throughout the paper we denote by χ the canonical additive character of $\mathbb{F}_q$. Recall that the orthogonality relation of the canonical additive character χ says that

$$\sum_{s\in\mathbb{F}_q} \chi(as) = \begin{cases} 0 & \text{if } a \neq 0 \\ q & \text{if } a = 0. \end{cases}$$

More generally, this implies that

$$\sum_{x\in\mathbb{F}_q^d} \chi(m\cdot x) = \begin{cases} 0 & \text{if } m \neq (0,\ldots,0) \\ q^d & \text{if } m = (0,\ldots,0), \end{cases}$$

where $\mathbb{F}_q^d$ denotes the d-dimensional vector space over $\mathbb{F}_q$ and $m\cdot x$ is the usual dot-product notation. Denote by $(\mathbb{F}_q^d, dx)$ the vector space over $\mathbb{F}_q$, endowed with the normalized counting measure "dx". Its dual space will be denoted by $(\mathbb{F}_q^d, dm)$ and we endow it with a counting measure "dm". If $f : (\mathbb{F}_q^d, dx) \to \mathbb{C}$, then the Fourier transform of the function f is defined on $(\mathbb{F}_q^d, dm)$:

$$\widehat{f}(m) = \int_{\mathbb{F}_q^d} f(x)\chi(-x\cdot m)\ dx = \frac{1}{q^d}\sum_{x\in\mathbb{F}_q^d} f(x)\chi(-x\cdot m) \quad \text{for } m \in \mathbb{F}_q^d.$$

We also recall that the Plancherel theorem yields that $\|\widehat{f}\|_{L^2(\mathbb{F}_q^d,dm)} = \|f\|_{L^2(\mathbb{F}_q^d,dx)}$ or

$$\sum_{m\in\mathbb{F}_q^d} |\widehat{f}(m)|^2 = \frac{1}{q^d}\sum_{x\in\mathbb{F}_q^d} |f(x)|^2.$$

For example, if f is a characteristic function on the subset E of $(\mathbb{F}_q^d, dx)$, then we see

$$\sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2 = \frac{|E|}{q^d}.$$

Here, and throughout the paper, we identify the set $E \subset \mathbb{F}_q^d$ with the characteristic function on the set E. We denote by $(d\sigma)^\vee$ the inverse Fourier transform of the normalized surface measure $d\sigma$ on S in (1.3) . Recall that

$$(d\sigma)^\vee(m) = \int_S \chi(m\cdot x)\ d\sigma(x) = \frac{1}{|S|}\sum_{x\in S}\chi(m\cdot x).$$

2.1. Gauss sums and estimates of $(d\sigma)^\vee$. Let η denote the quadratic character of $\mathbb{F}_q$. Recall that $\eta(t) = 1$ if s is a square number in $\mathbb{F}_q \setminus \{0\}$ and $\eta(t) = -1$ if t is not a square number in $\mathbb{F}_q \setminus \{0\}$. We also recall that $\eta(0) = 0, \eta^2 \equiv 1, \eta(ab) = \eta(a)\eta(b)$ for $a, b \in \mathbb{F}_q$, and $\eta(t) = \eta(t^{-1})$ for $t \neq 0$. For each $t \in \mathbb{F}_q$, the Gauss sum $G_t(\eta,\chi)$ is defined by

$$G_t(\eta,\chi) = \sum_{s\in\mathbb{F}_q\setminus\{0\}} \eta(s)\chi(ts).$$

Main properties of the Gauss sum are as follows (see Chapter 5 in [**7**] and Chapter 11 in [**2**]):

$$|G_t(\eta,\chi)| = \begin{cases} q^{\frac{1}{2}} & \text{if } t \neq 0 \\ 0 & \text{if } t = 0, \end{cases}$$

and

$$\sum_{s\in\mathbb{F}_q} \chi(ts^2) = \eta(t)G_1(\eta,\chi) \text{ for any } t \neq 0. \tag{2.1}$$

When we complete the square and apply a change of variables, it follows from the formula (2.1) that for each $a \in \mathbb{F}_q \setminus \{0\}, b \in \mathbb{F}_q$

$$\sum_{s \in \mathbb{F}_q} \chi(as^2 + bs) = G_1(\eta, \chi)\eta(a)\chi\left(\frac{b^2}{-4a}\right). \tag{2.2}$$

It turns out that the inverse Fourier transform of $d\sigma$ can be written in terms of the Gauss sum. The following lemma is due to the authors in [**4**]. For the readers' convenience, we introduce the proof .

LEMMA 2.1. *Let S be the variety in $\mathbb{F}_q^d$ as defined in (1.3), and let $d\sigma$ be the normalized surface measure on S. If $d \geq 2$ is even, then we have*

$$(d\sigma)^\vee(m) = \begin{cases} q^{d-1}|S|^{-1} + \frac{G_1^d}{|S|}(1 - q^{-1})\eta(a_1 \cdots a_d) & \text{if } m = (0, \ldots, 0) \\ \frac{G_1^d}{|S|}(1 - q^{-1})\eta(a_1 \cdots a_d) & \text{if } m \neq (0, \ldots, 0),\ \frac{m_1^2}{a_1} + \cdots + \frac{m_d^2}{a_d} = 0 \\ -\frac{G_1^d}{q|S|}\eta(a_1 \cdots a_d) & \text{if } \frac{m_1^2}{a_1} + \cdots + \frac{m_d^2}{a_d} \neq 0. \end{cases}$$

Here, and throughout this paper, we denote by η the quadratic character of $\mathbb{F}_q$ and we write G_1 for the Gauss sum $G_1(\eta, \chi)$.

PROOF. By the orthogonality relation of the canonical additive character χ of $\mathbb{F}_q$, we see

$$\begin{aligned}(d\sigma)^\vee(m) &= |S|^{-1} \sum_{x \in S} \chi(x \cdot m) = |S|^{-1} \sum_{x \in \mathbb{F}_q^d} \delta_0(a_1x_1^2 + \cdots + a_dx_d^2)\chi(x \cdot m) \\ &= |S|^{-1}q^{-1} \sum_{x \in \mathbb{F}_q^d} \sum_{s \in \mathbb{F}_q} \chi\left(s(a_1x_1^2 + \cdots + a_dx_d^2)\right)\ \chi(x \cdot m) \\ &= q^{d-1}|S|^{-1}\delta_0(m) + |S|^{-1}q^{-1} \sum_{x \in \mathbb{F}_q^d} \sum_{s \neq 0} \chi\left(s(a_1x_1^2 + \cdots + a_dx_d^2)\right)\ \chi(x \cdot m) \\ &= q^{d-1}|S|^{-1}\delta_0(m) + |S|^{-1}q^{-1} \sum_{s \neq 0} \prod_{j=1}^{d} \sum_{x_j \in \mathbb{F}_q} \chi(sa_jx_j^2 + m_jx_j).\end{aligned}$$

From the application of the inequality (2.2) we have

$$(d\sigma)^\vee(m) = q^{d-1}|S|^{-1}\delta_0(m) + G_1^d|S|^{-1}q^{-1}\eta(a_1 \cdots a_d) \sum_{s \neq 0} \eta^d(s)\chi\left(-\frac{1}{4s}\left(\frac{m_1^2}{a_1} + \cdots + \frac{m_d^2}{a_d}\right)\right).$$

Since $d \geq 2$ is even, it is clear that $\eta^d \equiv 1$. The proof is complete, because $\sum_{s \neq 0} \chi(as) = -1$ for all $a \neq 0$, and $\sum_{s \neq 0} \chi(as) = (q - 1)$ if $a = 0$. $\square$

Lemma 2.1 yields the following corollary.

COROLLARY 2.2. *Let S be the variety in $\mathbb{F}_q^d$ as defined in (1.3) and let $d\sigma$ be the normalized surface measure on S. If $d \geq 4$ is even, then we have*

$$|S| = q^{d-1} + G_1^d(1 - q^{-1})\eta(a_1 \cdots a_d) \sim q^{d-1}, \tag{2.3}$$

and

$$|(d\sigma)^\vee(m)| \sim \begin{cases} q^{-\frac{(d-2)}{2}} & \text{if } m \neq (0, \ldots, 0),\ \frac{m_1^2}{a_1} + \cdots + \frac{m_d^2}{a_d} = 0 \\ q^{-\frac{d}{2}} & \text{if } \frac{m_1^2}{a_1} + \cdots + \frac{m_d^2}{a_d} \neq 0. \end{cases} \tag{2.4}$$

PROOF. By the definition of $(d\sigma)^\vee(0,\dots,0)$, it is clear that $(d\sigma)^\vee(0,\dots,0)=1$. Comparison with Lemma 2.1 shows that

$$|S| = q^{d-1} + G_1^d(1-q^{-1})\eta(a_1\cdots a_d).$$

Since $|G_1| = q^{1/2}$, it follows that $|S|\sim q^{d-1}$ for $d\geq 4$ even. This proves (2.3). The inequality (2.4) follows immediately from Lemma 2.1 because $|G_1|=q^{1/2}$ and $|S|\sim q^{d-1}$ for $d\geq 4$ even. $\square$

REMARK 2.3. It is clear from (2.4) that if $d\geq 4$ is even and S is any nondegenerate quadratic surface in $\mathbb{F}_q^d$, then

$$|(d\sigma)^\vee(m)| = \left|\frac{1}{|S|}\sum_{x\in S}\chi(m\cdot x)\right| \sim \frac{1}{q^{d-1}}\left|\sum_{x\in S}\chi(m\cdot x)\right| \lesssim q^{-\frac{(d-2)}{2}} \text{ for } m\in\mathbb{F}_q^d\setminus\{(0,\dots,0)\}. \tag{2.5}$$

2.2. Bochner-Riesz kernel. Recall that $d\sigma$ is the normalized surface measure on the nondegenerate quadratic surface S . In the finite field setting, the Bochner-Riesz kernel K is a function on $(\mathbb{F}_q^d, dm)$ and it satisfies that $K=(d\sigma)^\vee-\delta_0$. Recall that dm denotes the counting measure on $\mathbb{F}_q^d$. Notice that $K(m)=0$ if $m=(0,\dots,0)$, and $K(m)=(d\sigma)^\vee(m)$ otherwise. Also observe that

$$d\sigma = \widehat{K}+\widehat{\delta_0} = \widehat{K}+1.$$

Here, the last equality follows because δ_0 is defined on the vector space with the counting measure dm, and its Fourier transform $\widehat{\delta_0}$ is defined on the dual space with the normalized counting measure dx. More precisely, if $x\in(\mathbb{F}_q^d, dx)$, then

$$\widehat{\delta_0}(x) = \int_{m\in\mathbb{F}_q^d}\chi(-m\cdot x)\delta_0(m)\ dm = \sum_{m\in\mathbb{F}_q^d}\chi(-m\cdot x)\delta_0(m) = 1.$$

Our main lemma is as follows.

LEMMA 2.4. *Suppose that $d\geq 6$ is even. Then, for every $E\subset\mathbb{F}_q^d$, we have*

$$\|E*\widehat{K}\|_{L^{\frac{d-1}{2}}(\mathbb{F}_q^d,dx)} \lesssim \begin{cases} q^{\frac{-d^2+2d-3}{d-1}}|E|^{\frac{d-3}{d-1}} & \text{if } 1\leq |E|\leq q^{\frac{d-2}{2}} \\ q^{\frac{-d^2+d-1}{d-1}}|E| & \text{if } q^{\frac{d-2}{2}}\leq |E|\leq q^{\frac{d}{2}} \\ q^{-d+1}|E|^{\frac{d-3}{d-1}} & \text{if } q^{\frac{d}{2}}\leq |E|\leq q^d. \end{cases} \tag{2.6}$$

where K is the Bochner-Riesz kernel. On the other hand, for every $E\subset\mathbb{F}_q^4$, it follows that

$$\|E*\widehat{K}\|_{L^6(\mathbb{F}_q^4,dx)} \lesssim \begin{cases} q^{-\frac{19}{6}}|E|^{\frac{5}{6}} & \text{if } 1\leq |E|\leq q \\ q^{-\frac{10}{3}}|E| & \text{if } q\leq |E|\leq q^2 \\ q^{-3}|E|^{\frac{5}{6}} & \text{if } q^2\leq |E|\leq q^4. \end{cases} \tag{2.7}$$

PROOF. Using the interpolation theorem, it suffices to prove that the following two inequalities hold for all d$\geq$ 4 even:

$$\|E*\widehat{K}\|_{L^\infty(\mathbb{F}_q^d,dx)} \lesssim q^{-d+1}|E| \tag{2.8}$$

and

$$\|E*\widehat{K}\|_{L^2(\mathbb{F}_q^d,dx)} \lesssim \begin{cases} q^{\frac{-2d+1}{2}}|E|^{\frac{1}{2}} & \text{if } 1\leq |E|\leq q^{\frac{d-2}{2}} \\ q^{\frac{-5d+4}{4}}|E| & \text{if } q^{\frac{d-2}{2}}\leq |E|\leq q^{\frac{d}{2}} \\ q^{-d+1}|E|^{\frac{1}{2}} & \text{if } q^{\frac{d}{2}}\leq |E|\leq q^d. \end{cases} \tag{2.9}$$

The estimate (2.8) can be obtained by applying Young's inequality. In fact, we see that

$$\|E*\widehat{K}\|_{L^\infty(\mathbb{F}_q^d,dx)} \leq \|\widehat{K}\|_{L^\infty(\mathbb{F}_q^d,dx)}\|E\|_{L^1(\mathbb{F}_q^d,dx)}.$$

Since $\|\widehat{K}\|_{L^\infty(\mathbb{F}_q^d,dx)} \lesssim q$ and $\|E\|_{L^1(\mathbb{F}_q^d,dx)} = q^{-d}|E|$, the inequality (2.8) is established. To prove the inequality (2.9), first use the Plancherel theorem. It follows that

$$\|E * \widehat{K}\|^2_{L^2(\mathbb{F}_q^d,dx)} = \|\widehat{E}K\|^2_{L^2(\mathbb{F}_q^d,dm)}.$$

Now, we recall that dx is the normalized counting measure but dm is the counting measure. Thus, the expression above is given by

$$\begin{aligned}\sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2|K(m)|^2 &= \sum_{m\neq(0,\dots,0)} |\widehat{E}(m)|^2|(d\sigma)^\vee(m)|^2\\ &\sim \frac{1}{q^{d-2}} \sum_{\substack{m\neq(0,\dots,0):\\ \frac{m_1^2}{a_1}+\cdots+\frac{m_d^2}{a_d}=0}} |\widehat{E}(m)|^2 + \frac{1}{q^d} \sum_{\substack{m\neq(0,\dots,0):\\ \frac{m_1^2}{a_1}+\cdots+\frac{m_d^2}{a_d}\neq0}} |\widehat{E}(m)|^2 = \mathrm{I}+\mathrm{II},\end{aligned}$$

where the first line and the second line follow from the definition of K and the inequality (2.4) in Corollary 2.2 respectively. Applying the Plancherel theorem, it is clear that

$$\mathrm{II} \le \frac{1}{q^d}\sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2 = q^{-2d}|E|. \tag{2.10}$$

In order to obtain a good upper bound of I, we shall conduct two different estimates on I. First, the Plancherel theorem yields

$$\mathrm{I} \le \frac{1}{q^{d-2}}\sum_{m\in\mathbb{F}_q^d} |\widehat{E}(m)|^2 = \frac{|E|}{q^{2d-2}}. \tag{2.11}$$

On the other hand, it follows that

$$\mathrm{I} \le \frac{1}{q^{d-2}} \sum_{\frac{m_1^2}{a_1}+\cdots+\frac{m_d^2}{a_d}=0} |\widehat{E}(m)|^2 = \frac{1}{q^{3d-2}} \sum_{\frac{m_1^2}{a_1}+\cdots+\frac{m_d^2}{a_d}=0} \sum_{x,y\in E} \chi(-m\cdot(x-y)).$$

Now, let $S_a = \{m \in \mathbb{F}_q^d : \frac{m_1^2}{a_1}+\cdots+\frac{m_d^2}{a_d}=0\}$ which is also a nondegenerate quadratic surface with $|S_a| \sim q^{d-1}$. Then the expression above can be written by

$$\frac{1}{q^{3d-2}} \sum_{x,y\in E: x=y} |S_a| + \frac{1}{q^{3d-2}} \sum_{x,y\in E:x\neq y} \left(\sum_{m\in S_a} \chi(-m\cdot(x-y))\right).$$

Now, we see from (2.5) that if $x \neq y$, then $\left|\sum_{m\in S_a}\chi(-m\cdot(x-y))\right| \lesssim q^{\frac{d}{2}}$. Thus, we obtain that

$$\mathrm{I} \lesssim q^{-2d+1}|E| + q^{\frac{-5d+4}{2}}|E|^2.$$

Combining this with the inequality (2.11) gives

$$\mathrm{I} \lesssim \min\left(\frac{|E|}{q^{2d-2}},\ q^{-2d+1}|E| + q^{\frac{-5d+4}{2}}|E|^2\right).$$

In conjunction with the inequality (2.10), this shows that

$$\|E*\widehat{K}\|^2_{L^2(\mathbb{F}_q^d,dx)} \lesssim \min\left(\frac{|E|}{q^{2d-2}},\ q^{-2d+1}|E| + q^{\frac{-5d+4}{2}}|E|^2\right) + q^{-2d}|E|.$$

Since $(\alpha+\beta)^{1/2} \sim \alpha^{1/2}+\beta^{1/2}$ for $\alpha,\beta \ge 0$, it also follows that

$$\|E*\widehat{K}\|_{L^2(\mathbb{F}_q^d,dx)} \lesssim \min\left(q^{-d+1}|E|^{\frac12},\ q^{\frac{-2d+1}{2}}|E|^{\frac12} + q^{\frac{-5d+4}{4}}|E|\right) + q^{-d}|E|^{\frac12}.$$

A direct computation shows that this implies the inequality (2.9). We complete the proof of Lemma 2.4.

$\square$

3. Proof of Theorem 1.2 for $d \geq 6$

In this section we provide the complete proof of Theorem 1.2 in the case that $d \geq 6$ is even. The proof for $d = 4$ shall be independently given in the following section. The main reason is as follows. Lemma 2.4 shall be used to prove Theorem 1.2. If $d \geq 6$ is even, then we have seen that the inequality (2.6) of Lemma 2.4 follows by interpolating (2.8) and (2.9). However, if d is four, then such an interpolation is too meaningless to assert that (2.6) holds for $d = 4$. As an alternative approach, the inequality (2.7) of Lemma 2.4 shall be applied to complete the proof for $d = 4$. In this case we need more delicate estimates.

Now we start proving Theorem 1.2 for $d \geq 6$ even. As mentioned in Remark 1.4, it is enough to prove the following statement.

THEOREM 3.1. *Let S be the variety in $\mathbb{F}_q^d$ as defined in (1.3). If $d \geq 6$ is even, then we have*

$$\|f * d\sigma\|_{L^r(\mathbb{F}_q^d, dx)} \lesssim \|f\|_{L^p(\mathbb{F}_q^d, dx)} \text{ for all functions } f \text{ on } \mathbb{F}_q^d,$$

where $(p, r) = \left(\frac{d^2-d}{d^2-2d+2}, d-1\right)$.

PROOF. Let $p = \frac{d^2-d}{d^2-2d+2}$ and $r = d-1$. We aim to prove that for every complex-valued function f on $\mathbb{F}_q^d$,

$$\|f * d\sigma\|_{L^r(\mathbb{F}_q^d, dx)} \lesssim \|f\|_{L^p(\mathbb{F}_q^d, dx)} = \left(q^{-d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^p\right)^{\frac{1}{p}}.$$

As in [**6**] we proceed with the proof by decomposing the function f to which the operator is applied into level sets. Without loss of generality, we may assume that f is a nonnegative real, simple function given by the form

$$f = \sum_{k=0}^{\infty} 2^{-k} E_k, \tag{3.1}$$

where $E_0, E_1, \ldots$ are disjoint subsets of $\mathbb{F}_q^d$. Also notice that we may assume that

$$\sum_{x \in \mathbb{F}_q^d} |f(x)|^p = 1.$$

It follows from these assumptions that

$$\sum_{j=0}^{\infty} 2^{-pj} |E_j| = 1, \tag{3.2}$$

and hence for every $j = 0, 1, \ldots,$

$$|E_j| \leq 2^{pj}. \tag{3.3}$$

Recall that $d\sigma = \widehat{K} + 1$ where K is the Bochner-Riesz kernel. It follows that

$$\|f * d\sigma\|_{L^r(\mathbb{F}_q^d, dx)} \leq \|f * \widehat{K}\|_{L^r(\mathbb{F}_q^d, dx)} + \|f * 1\|_{L^r(\mathbb{F}_q^d, dx)}.$$

Since $r > p$ and dx is the normalized counting measure on $\mathbb{F}_q^d$, it is clear from Young's inequality that

$$\|f * 1\|_{L^r(\mathbb{F}_q^d, dx)} \leq \|f\|_{L^p(\mathbb{F}_q^d, dx)}.$$

Therefore, it suffices to prove the following inequality

$$\|f * \widehat{K}\|_{L^r(\mathbb{F}_q^d,dx)} \lesssim \|f\|_{L^p(\mathbb{F}_q^d,dx)}.$$

Since we have assumed that $\sum_{x\in\mathbb{F}_q^d}|f(x)|^p = 1$, we see that $\|f\|_{L^p(\mathbb{F}_q^d,dx)} = q^{-d/p}$. Also observe that

$$\|f * \widehat{K}\|^2_{L^r(\mathbb{F}_q^d,dx)} = \|(f * \widehat{K})(f * \widehat{K})\|_{L^{\frac{r}{2}}(\mathbb{F}_q^d,dx)}.$$

From these observations, our task is to show that

$$q^{\frac{2d}{p}}\|(f * \widehat{K})(f * \widehat{K})\|_{L^{\frac{r}{2}}(\mathbb{F}_q^d,dx)} \lesssim 1. \tag{3.4}$$

Since f is the simple function in (3.1), we see that

$$\begin{aligned} q^{\frac{2d}{p}}\|(f * \widehat{K})(f * \widehat{K})\|_{L^{\frac{r}{2}}(\mathbb{F}_q^d,dx)} &\leq q^{\frac{2d}{p}}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}2^{-k-j}\|(E_k * \widehat{K})(E_j * \widehat{K})\|_{L^{\frac{r}{2}}(\mathbb{F}_q^d,dx)} \\ &\lesssim q^{\frac{2d}{p}}q^{-d+1}\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}2^{-k-j}|E_k|\|(E_j * \widehat{K})\|_{L^{\frac{r}{2}}(\mathbb{F}_q^d,dx)}, \end{aligned}$$

where the last line follows by the symmetry of k and j, and the inequality (2.8). Now, for each $j = 0, 1, 2, \ldots,$ we consider the following three sets:

$$J_1 = \{j : 1 \leq |E_j| \leq q^{\frac{d-2}{2}}\},$$
$$J_2 = \{j : q^{\frac{d-2}{2}} < |E_j| \leq q^{\frac{d}{2}}\},$$

and

$$J_3 = \{j : q^{\frac{d}{2}} < |E_j| \leq q^{d}\}.$$

Since $r/2 = (d-1)/2$, it is clear from (2.6) in Lemma 2.4 that our goal is to prove the following three inequalities:

$$A_1 := q^{\frac{2d}{p}}q^{-d+1}q^{\frac{-d^2+2d-3}{d-1}}\sum_{k=0}^{\infty}\sum_{j=k:j\in J_1}^{\infty}2^{-k-j}|E_k||E_j|^{\frac{d-3}{d-1}} \lesssim 1, \tag{3.5}$$

$$A_2 := q^{\frac{2d}{p}}q^{-d+1}q^{\frac{-d^2+d-1}{d-1}}\sum_{k=0}^{\infty}\sum_{j=k:j\in J_2}^{\infty}2^{-k-j}|E_k||E_j| \lesssim 1, \tag{3.6}$$

$$A_3 := q^{\frac{2d}{p}}q^{-d+1}q^{-d+1}\sum_{k=0}^{\infty}\sum_{j=k:j\in J_3}^{\infty}2^{-k-j}|E_k||E_j|^{\frac{d-3}{d-1}} \lesssim 1. \tag{3.7}$$

First, we prove that the inequality (3.5) holds. Since $p = \frac{d^2-d}{d^2-2d+2}$, a direct computation shows that $q^{\frac{2d}{p}}q^{-d+1}q^{\frac{-d^2+2d-3}{d-1}} = 1$. Now recall from (3.3) that $|E_j| \leq 2^{pj}$ for all $j = 0, 1, \ldots$. Therefore, it follows that

$$A_1 \leq \sum_{k=0}^{\infty}\sum_{j=k:j\in J_1}^{\infty}2^{-k-j}|E_k|2^{\frac{jp(d-3)}{d-1}} \leq \sum_{k=0}^{\infty}2^{-k}|E_k|\sum_{j=k}^{\infty}2^{j\left(-1+\frac{p(d-3)}{d-1}\right)}.$$

Since $-1+\frac{p(d-3)}{d-1} = \frac{-d-2}{d^2-2d+2} < 0$ and the sum over j is a geometric series, we see that $\sum_{j=k}^{\infty}2^{j\left(-1+\frac{p(d-3)}{d-1}\right)} \sim 2^{k\left(-1+\frac{p(d-3)}{d-1}\right)}$. Thus, the inequality (3.5) is established as follows:

$$A_1 \lesssim \sum_{k=0}^{\infty}|E_k|2^{k\left(-2+\frac{p(d-3)}{d-1}\right)} \leq \sum_{k=0}^{\infty}|E_k|2^{-pk} = 1,$$

where we used the simple observation that $-2+\frac{p(d-3)}{d-1}<-p$, and then the assumption (3.2). Second, we prove that the inequality (3.6) holds. Let $\varepsilon=\frac{2d-4}{d^2-d}$. Since $d\geq 6$, we see that $0<\varepsilon<1$. Write A_2 as follows:

$$A_2=q^{\frac{2d}{p}}q^{-d+1}q^{\frac{-d^2+d-1}{d-1}}\sum_{k=0}^{\infty}\sum_{j=k:j\in J_2}^{\infty}2^{-k-j}|E_k||E_j|^{1-\varepsilon}|E_j|^{\varepsilon}.$$

Since $0<\varepsilon<1$, we notice from (3.3) that $|E_j|^{1-\varepsilon}\leq 2^{p(1-\varepsilon)j}$. By the definition of the set J_2, we also see that $|E_j|^{\varepsilon}\leq q^{\frac{d\varepsilon}{2}}$ for all $j\in J_2$. Then, we have

$$\begin{aligned}A_2&\leq q^{\frac{2d}{p}}q^{-d+1}q^{\frac{-d^2+d-1}{d-1}}q^{\frac{d\varepsilon}{2}}\sum_{k=0}^{\infty}\sum_{j=k:j\in J_2}^{\infty}2^{-k-j}|E_k|2^{p(1-\varepsilon)j}\\&\leq q^{\frac{2d}{p}}q^{-d+1}q^{\frac{-d^2+d-1}{d-1}}q^{\frac{d\varepsilon}{2}}\sum_{k=0}^{\infty}2^{-k}|E_k|\sum_{j=k}^{\infty}2^{j(-1+p(1-\varepsilon))}.\end{aligned}$$

Notice that $q^{\frac{2d}{p}}q^{-d+1}q^{\frac{-d^2+d-1}{d-1}}q^{\frac{d\varepsilon}{2}}=1$, and the geometric series over j converges to $\sim 2^{k(-1+p(1-\varepsilon))}$ because $-1+p(1-\varepsilon)=\frac{-d+2}{d^2-2d+2}<0$ for $d\geq 6$. From this observation and (3.2), the inequality (3.6) follows because we have

$$A_2\lesssim\sum_{k=0}^{\infty}|E_k|2^{k(-2+p(1-\varepsilon))}=\sum_{k=0}^{\infty}|E_k|2^{-pk}=1.$$

Finally, we show that the inequality (3.7) holds. As in the proof of the inequality (3.6), we let $0<\delta=\frac{4}{d^2-d}<1$ for $d\geq 6$. The value A_3 is written by

$$A_3=q^{\frac{2d}{p}}q^{-d+1}q^{-d+1}\sum_{k=0}^{\infty}\sum_{j=k:j\in J_3}^{\infty}2^{-k-j}|E_k||E_j|^{\delta+\frac{d-3}{d-1}}|E_j|^{-\delta}.$$

Notice from (3.3) that $|E_j|^{\delta+\frac{d-3}{d-1}}\leq 2^{p\left(\delta+\frac{d-3}{d-1}\right)j}$ for all $j=0,1,2,\ldots$. By the definition of J_3, it is easy to notice that $|E_j|^{-\delta}\leq q^{\frac{-d\delta}{2}}$ for $j\in J_3$. It therefore follows that

$$\begin{aligned}A_3&\leq q^{\frac{2d}{p}}q^{-d+1}q^{-d+1}q^{\frac{-d\delta}{2}}\sum_{k=0}^{\infty}\sum_{j=k:j\in J_3}^{\infty}2^{-k-j}|E_k|2^{p\left(\delta+\frac{d-3}{d-1}\right)j}\\&\leq\sum_{k=0}^{\infty}2^{-k}|E_k|\sum_{j=k}^{\infty}2^{\left(-1+p\left(\delta+\frac{d-3}{d-1}\right)\right)j}\\&\sim\sum_{k=0}^{\infty}|E_k|2^{\left(-2+p\left(\delta+\frac{d-3}{d-1}\right)\right)k}=\sum_{k=0}^{\infty}|E_k|2^{-pk}=1,\end{aligned}$$

where we used the facts that $q^{\frac{2d}{p}}q^{-d+1}q^{-d+1}q^{\frac{-d\delta}{2}}=1$, $\left(-1+p\left(\delta+\frac{d-3}{d-1}\right)\right)=\frac{-d+2}{d^2-2d+2}<0$ for $d\geq 6$, and $\left(-2+p\left(\delta+\frac{d-3}{d-1}\right)\right)=-p$, and then the assumption (3.2) for the last equality. Thus, the inequality (3.7) holds and the proof of Theorem 3.1 is complete. □

4. Proof of Theorem 1.2 for $d=4$

As observed in Remark 1.4, it amounts to showing the following statement.

Theorem 4.1. *Let S be the variety in $\mathbb{F}_q^4$ as defined in (1.3). Then, we have*

$$\|f*d\sigma\|_{L^3(\mathbb{F}_q^4,dx)} \lesssim \|f\|_{L^{\frac{6}{5}}(\mathbb{F}_q^4,dx)} \text{ for all functions } f \text{ on } \mathbb{F}_q^4.$$

Proof. We will proceed by the similar ways as in the previous section. However, the proof of the theorem will be based on (2.7), rather than (2.6) in Lemma 2.4. We begin by recalling from (2.7) and (2.9) that

$$\|E*\widehat{K}\|_{L^6(\mathbb{F}_q^4,dx)} \lesssim \begin{cases} q^{-\frac{19}{6}}|E|^{\frac{5}{6}} & \text{if } 1\le |E|\le q\\ q^{-\frac{10}{3}}|E| & \text{if } q\le |E|\le q^2\\ q^{-3}|E|^{\frac{5}{6}} & \text{if } q^2\le |E|\le q^4,\end{cases} \tag{4.1}$$

and

$$\|E*\widehat{K}\|_{L^2(\mathbb{F}_q^4,dx)} \lesssim \begin{cases} q^{-\frac{7}{2}}|E|^{\frac{1}{2}} & \text{if } 1\le |E|\le q\\ q^{-4}|E| & \text{if } q\le |E|\le q^2\\ q^{-3}|E|^{\frac{1}{2}} & \text{if } q^2\le |E|\le q^4.\end{cases} \tag{4.2}$$

We must show that for all complex-valued functions f on $\mathbb{F}_q^4$,

$$\|f*d\sigma\|_{L^3(\mathbb{F}_q^4,dx)} \lesssim \|f\|_{L^{\frac{6}{5}}(\mathbb{F}_q^4,dx)}.$$

As noticed in the previous section, it suffices to prove this inequality under the following assumptions:

$$\sum_{x\in\mathbb{F}_q^4}|f(x)|^{\frac{6}{5}}=1 \quad\text{and}\quad f=\sum_{k=0}^{\infty}2^{-k}E_k,$$

where $E_0,E_1,\dots$ are disjoint subsets of $\mathbb{F}_q^4$. From these assumptions, it is clear that

$$\sum_{j=0}^{\infty}2^{-\frac{6j}{5}}|E_j|=1 \quad \text{for all } j=0,1,\dots. \tag{4.3}$$

This clearly implies that

$$|E_j|\le 2^{\frac{6j}{5}} \quad \text{for all } j=0,1,\dots. \tag{4.4}$$

According to (3.4), it is enough to prove that

$$q^{\frac{20}{3}}\|(f*\widehat{K})(f*\widehat{K})\|_{L^{\frac{3}{2}}(\mathbb{F}_q^4,dx)}\lesssim 1.$$

Since $f=\sum_{k=0}^{\infty}2^{-k}E_k$, it is enough to show that

$$q^{\frac{20}{3}}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}2^{-k-j}\|(E_k*\widehat{K})(E_j*\widehat{K})\|_{L^{\frac{3}{2}}(\mathbb{F}_q^4,dx)}\lesssim 1.$$

By the symmetry of k and j, and the Hölder inequality, our task is to prove

$$q^{\frac{20}{3}}\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}2^{-k-j}\|(E_k*\widehat{K})\|_{L^6(\mathbb{F}_q^4,dx)}\|(E_j*\widehat{K})\|_{L^2(\mathbb{F}_q^4,dx)}\lesssim 1.$$

Main steps to prove this inequality are summarized as follows. By considering the sizes of $|E_k|$ and $|E_j|$, we first decompose $\sum_{k=0}^{\infty}\sum_{j=k}^{\infty}$ as nine parts. Next, using the estimates (4.1),(4.2), (4.4), (4.3), and a convergence property of a geometric series, we show that each part of them is $\lesssim 1$, which completes the proof of Theorem 4.1. For the sake of completeness, we shall give full details. To do this, let us define the following 9 sets: for $N=\{0,1,\dots\}$,

$$I_1=\{(k,j)\in N\times N : k\le j,\ 1\le |E_k|\le q,\ 1\le |E_j|\le q\},$$

$$I_2 = \{(k,j) \in N \times N : k \le j,\ 1 \le |E_k| \le q,\ q < |E_j| \le q^2\},$$
$$I_3 = \{(k,j) \in N \times N : k \le j,\ 1 \le |E_k| \le q,\ q^2 < |E_j| \le q^4\},$$
$$I_4 = \{(k,j) \in N \times N : k \le j,\ q < |E_k| \le q^2,\ 1 \le |E_j| \le q\},$$
$$I_5 = \{(k,j) \in N \times N : k \le j,\ q < |E_k| \le q^2,\ q < |E_j| \le q^2\},$$
$$I_6 = \{(k,j) \in N \times N : k \le j,\ q < |E_k| \le q^2,\ q^2 < |E_j| \le q^4\},$$
$$I_7 = \{(k,j) \in N \times N : k \le j,\ q^2 < |E_k| \le q^4,\ 1 \le |E_j| \le q\},$$
$$I_8 = \{(k,j) \in N \times N : k \le j,\ q^2 < |E_k| \le q^4,\ q < |E_j| \le q^2\},$$
$$I_9 = \{(k,j) \in N \times N : k \le j,\ q^2 < |E_k| \le q^4,\ q^2 < |E_j| \le q^4\}.$$

4.1. Estimate of the sum over I_1. It follows from (4.1) and (4.2) that

$$\begin{aligned}
&q^{\frac{20}{3}} \sum_{(k,j)\in I_1} 2^{-k-j} \|(E_k * \widehat{K})\|_{L^6(\mathbb{F}_q^4,dx)} \|(E_j * \widehat{K})\|_{L^2(\mathbb{F}_q^4,dx)} \\
&\lesssim \sum_{(k,j)\in I_1} 2^{-k-j} |E_k|^{\frac{5}{6}} |E_j|^{\frac{1}{2}} \le \sum_{(k,j)\in I_1} 2^{-k-j} |E_k| 2^{\frac{3j}{5}} \quad \text{since} \quad |E_j|^{\frac{1}{2}} \le 2^{\frac{3j}{5}} \quad \text{by (4.4)} \\
&\le \sum_{k=0}^{\infty} |E_k| 2^{-k} \sum_{j=k}^{\infty} 2^{-\frac{2j}{5}} \sim \sum_{k=0}^{\infty} |E_k| 2^{-\frac{7k}{5}} \le \sum_{k=0}^{\infty} 2^{-\frac{6k}{5}} |E_k| = 1 \quad \text{by (4.3).}
\end{aligned}$$

4.2. Estimate of the sum over I_2. It follows from (4.1) and (4.2) that

$$\begin{aligned}
&q^{\frac{20}{3}} \sum_{(k,j)\in I_2} 2^{-k-j} \|(E_k * \widehat{K})\|_{L^6(\mathbb{F}_q^4,dx)} \|(E_j * \widehat{K})\|_{L^2(\mathbb{F}_q^4,dx)} \\
&\lesssim q^{-\frac{1}{2}} \sum_{(k,j)\in I_2} 2^{-k-j} |E_k|^{\frac{5}{6}} |E_j| \le q^{-\frac{1}{2}} \sum_{(k,j)\in I_2} 2^{-j} |E_j| \quad \text{since} \quad 2^{-k} |E_k|^{\frac{5}{6}} \le 1 \quad \text{by (4.4)} \\
&\le \sum_{(k,j)\in I_2} 2^{-j} |E_j|^{\frac{3}{4}} \quad \text{since} \quad |E_j|^{\frac{1}{4}} \le q^{\frac{1}{2}} \quad \text{for } (k,j) \in I_2 \\
&\le \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-\frac{j}{10}} \quad \text{by (4.4)} \\
&\sim \sum_{k=0}^{\infty} 2^{-\frac{k}{10}} \sim 1.
\end{aligned}$$

4.3. Estimate of the sum over I_3. It follows from (4.1) and (4.2) that

$$\begin{aligned}
&q^{\frac{20}{3}} \sum_{(k,j)\in I_3} 2^{-k-j} \|(E_k * \widehat{K})\|_{L^6(\mathbb{F}_q^4,dx)} \|(E_j * \widehat{K})\|_{L^2(\mathbb{F}_q^4,dx)} \\
&\lesssim q^{\frac{1}{2}} \sum_{(k,j)\in I_3} 2^{-k-j} |E_k|^{\frac{5}{6}} |E_j|^{\frac{1}{2}} \le q^{\frac{1}{2}} \sum_{(k,j)\in I_3} 2^{-j} |E_j|^{\frac{1}{2}} \quad \text{by (4.4)} \\
&= \sum_{(k,j)\in I_3} 2^{-j} |E_j|^{\frac{3}{4}} q^{\frac{1}{2}} |E_j|^{-\frac{1}{4}} < \sum_{(k,j)\in I_3} 2^{-j} |E_j|^{\frac{3}{4}} \quad \text{since} \quad q^2 < |E_j| \text{ for } (k,j) \in I_3 \\
&\le \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} 2^{-\frac{j}{10}} \sim \sum_{k=0}^{\infty} 2^{-\frac{k}{10}} \sim 1.
\end{aligned}$$

where (4.4) was also used to obtain the last inequality .

4.4. Estimate of the sum over I_4. It follows from (4.1) and (4.2) that

$$
\begin{aligned}
& q^{\frac{20}{3}} \sum_{(k,j)\in I_4} 2^{-k-j} \|(E_k * \widehat{K})\|_{L^6(\mathbb{F}_q^4,dx)} \|(E_j * \widehat{K})\|_{L^2(\mathbb{F}_q^4,dx)} \\
\lesssim & q^{-\frac{1}{6}} \sum_{(k,j)\in I_4} 2^{-k-j}|E_k||E_j|^{\frac{1}{2}} \le q^{-\frac{1}{6}} \sum_{(k,j)\in I_4} 2^{-k}|E_k|2^{-j}2^{\frac{3j}{5}} \quad \text{by (4.4)} \\
\le & q^{-\frac{1}{6}} \sum_{k=0}^{\infty} 2^{-k}|E_k| \sum_{j=k}^{\infty} 2^{-\frac{2j}{5}} \sim q^{-\frac{1}{6}} \sum_{k=0}^{\infty} 2^{-\frac{7k}{5}}|E_k| \\
\le & \sum_{k=0}^{\infty} 2^{-\frac{6k}{5}}|E_k| = 1 \quad \text{by (4.3).}
\end{aligned}
$$

4.5. Estimate of the sum over I_5. It follows from (4.1) and (4.2) that

$$
\begin{aligned}
& q^{\frac{20}{3}} \sum_{(k,j)\in I_5} 2^{-k-j} \|(E_k * \widehat{K})\|_{L^6(\mathbb{F}_q^4,dx)} \|(E_j * \widehat{K})\|_{L^2(\mathbb{F}_q^4,dx)} \\
\lesssim & q^{-\frac{2}{3}} \sum_{(k,j)\in I_5} 2^{-k-j}|E_k||E_j| \le \sum_{(k,j)\in I_5} 2^{-k-j}|E_k||E_j|^{\frac{2}{3}} \quad \text{since } |E_j|^{\frac{1}{3}} \le q^{\frac{2}{3}} \quad \text{for } (k,j)\in I_5 \\
\le & \sum_{k=0}^{\infty} 2^{-k}|E_k| \sum_{j=k}^{\infty} 2^{-\frac{j}{5}} \sim \sum_{k=0}^{\infty} 2^{-\frac{6k}{5}}|E_k| = 1,
\end{aligned}
$$

where we used (4.4), the convergence of a geometric series, and (4.3) in the last line.

4.6. Estimate of the sum over I_6. It follows from (4.1) and (4.2) that

$$
\begin{aligned}
& q^{\frac{20}{3}} \sum_{(k,j)\in I_6} 2^{-k-j} \|(E_k * \widehat{K})\|_{L^6(\mathbb{F}_q^4,dx)} \|(E_j * \widehat{K})\|_{L^2(\mathbb{F}_q^4,dx)} \\
\lesssim & q^{\frac{1}{3}} \sum_{(k,j)\in I_6} 2^{-k-j}|E_k||E_j|^{\frac{1}{2}} < \sum_{(k,j)\in I_6} 2^{-k-j}|E_k||E_j|^{\frac{1}{2}+\frac{1}{6}} \quad \text{since } |E_j|^{-\frac{1}{6}} < q^{-\frac{1}{3}} \quad \text{for } (k,j)\in I_6 \\
\le & \sum_{k=0}^{\infty} 2^{-k}|E_k| \sum_{j=k}^{\infty} 2^{-\frac{j}{5}} \sim \sum_{k=0}^{\infty} 2^{-\frac{6k}{5}}|E_k| = 1,
\end{aligned}
$$

where (4.4), the convergence of a geometric series, and (4.3) were also applied in the last line.

4.7. Estimate of the sum over I_7. It follows from (4.1) and (4.2) that

$$
\begin{aligned}
& q^{\frac{20}{3}} \sum_{(k,j)\in I_7} 2^{-k-j} \|(E_k * \widehat{K})\|_{L^6(\mathbb{F}_q^4,dx)} \|(E_j * \widehat{K})\|_{L^2(\mathbb{F}_q^4,dx)} \\
\lesssim & q^{\frac{1}{6}} \sum_{(k,j)\in I_7} 2^{-k-j}|E_k|^{\frac{5}{6}}|E_j|^{\frac{1}{2}} = q^{\frac{1}{6}} \sum_{(k,j)\in I_7} 2^{-k-j}|E_k||E_k|^{-\frac{1}{6}}|E_j|^{\frac{1}{2}} \\
< & q^{\frac{1}{6}-\frac{1}{3}} \sum_{(k,j)\in I_7} 2^{-k}|E_k|2^{-\frac{2j}{5}} \quad \text{by observing} \quad |E_k|^{-\frac{1}{6}} < q^{-\frac{1}{3}} \quad \textit{for } (k,j)\in I_7 \text{ and by (4.4)} \\
\le & \sum_{k=0}^{\infty} 2^{-k}|E_k| \sum_{j=k}^{\infty} 2^{-\frac{2j}{5}} \sim \sum_{k=0}^{\infty} 2^{-\frac{7k}{5}}|E_k| \le \sum_{k=0}^{\infty} 2^{-\frac{6k}{5}}|E_k| = 1.
\end{aligned}
$$

4.8. Estimate of the sum over I_8. It follows from (4.1) and (4.2) that

$$
\begin{aligned}
&q^{\frac{20}{3}} \sum_{(k,j)\in I_8} 2^{-k-j} \|(E_k * \widehat{K})\|_{L^6(\mathbb{F}_q^4,dx)} \|(E_j * \widehat{K})\|_{L^2(\mathbb{F}_q^4,dx)} \\
&\lesssim q^{-\frac{1}{3}} \sum_{(k,j)\in I_8} 2^{-k-j} |E_k|^{\frac{5}{6}} |E_j| = q^{-\frac{1}{3}} \sum_{(k,j)\in I_8} 2^{-k-j} |E_k||E_j|^{\frac{2}{3}} |E_k|^{-\frac{1}{6}} |E_j|^{\frac{1}{3}} \\
&< \sum_{(k,j)\in I_8} 2^{-k-j} |E_k||E_j|^{\frac{2}{3}} \quad \text{since } |E_k|^{-\frac{1}{6}} < q^{-\frac{1}{3}}, \quad |E_j|^{\frac{1}{3}} \le q^{\frac{2}{3}} \text{ for } (k,j) \in I_8 \\
&\le \sum_{k=0}^{\infty} 2^{-k} |E_k| \sum_{j=k}^{\infty} 2^{-\frac{j}{5}} \sim \sum_{k=0}^{\infty} 2^{-\frac{6k}{5}} |E_k| = 1.
\end{aligned}
$$

4.9. Estimate of the sum over I_9. It follows from (4.1) and (4.2) that

$$
\begin{aligned}
&q^{\frac{20}{3}} \sum_{(k,j)\in I_9} 2^{-k-j} \|(E_k * \widehat{K})\|_{L^6(\mathbb{F}_q^4,dx)} \|(E_j * \widehat{K})\|_{L^2(\mathbb{F}_q^4,dx)} \\
&\lesssim q^{\frac{2}{3}} \sum_{(k,j)\in I_9} 2^{-k-j} |E_k|^{\frac{5}{6}} |E_j|^{\frac{1}{2}} = q^{\frac{2}{3}} \sum_{(k,j)\in I_9} 2^{-k-j} |E_k||E_j|^{\frac{1}{2}+\frac{1}{6}} |E_k|^{-\frac{1}{6}} |E_j|^{-\frac{1}{6}} \\
&< \sum_{(k,j)\in I_9} 2^{-k-j} |E_k||E_j|^{\frac{1}{2}+\frac{1}{6}} \quad \text{since } |E_k|^{-\frac{1}{6}}, |E_j|^{-\frac{1}{6}} < q^{-\frac{1}{3}} \text{ for } (k,j) \in I_9 \\
&= \sum_{k=0}^{\infty} 2^{-k} |E_k| \sum_{j=k}^{\infty} 2^{-j} |E_j|^{\frac{2}{3}} \le \sum_{k=0}^{\infty} 2^{-k} |E_k| \sum_{j=k}^{\infty} 2^{-\frac{j}{5}} \sim \sum_{k=0}^{\infty} 2^{-\frac{6k}{5}} |E_k| = 1,
\end{aligned}
$$

where we also used (4.4), the convergence of a geometric series, and (4.3) in the last line. □

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